STATEMENT OF RESEARCH INTERESTS

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1. Overview

My research is in number theory, with particular interest in modular forms and their applications. Roughly speaking, a modular form $f(\tau)$ is a complex-valued function defined on the upper-half of the complex plane containing a vast amount of symmetry. Specifically, we ask that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \tag{1}$$

for all τ in the upper-half plane, some fixed positive integer (or half-integer) k, and all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in either the group $\mathrm{SL}_2(\mathbb{Z})$ of 2×2 matrices with integer coefficients and determinant 1 or some subgroup thereof. The transformation property (1) may appear restrictive enough to prevent any such functions from existing at all, but in fact they are seemingly ubiquitous throughout mathematics. Although originally developed for and primarily used in number theory—most famously in Wiles' proof of Fermat's Last Theorem [33]—these functions have been used to study topics as disparate as knots [19], black holes [11, 34], and homotopy groups of spheres [20]. These widespread applications inform a large part of my interest in modular forms—it can be easy as a theoretical mathematician to fall deeper into our own specializations and lose sight of a broader view of mathematics, but modular forms give an easy gateway into projects and ideas outside of my primary areas of focus. For example, in §3, I discuss an ongoing project connecting modular forms and colored Jones polynomials of knots, and §4 concerns work on quadratic class numbers by viewing them as Fourier coefficients of modular forms.

A central point of focus in my research is in hypergeometric functions and their relationship to modular forms. The hypergeometric function with parameters $\alpha = \{r_1, r_2, \dots, r_n\}$ and $\beta = \{q_1, q_2, \dots, q_{n-1}\}$ is defined as

$${}_nF_{n-1}\left[\begin{matrix} r_1 & r_2 & \dots & r_n \\ & q_1 & \dots & q_{n-1} \end{matrix} \; ; \; z\right] := \sum_{k=0}^{\infty} \frac{(r_1)_k (r_2)_k \dots (r_n)_k}{(q_1)_k (q_2)_k \dots (q_{n-1})_k} \frac{z^k}{k!}, \tag{2}$$

where $(a)_k$ denotes the *rising factorial* or *Pochhammer symbol*

$$(a)_k = a(a+1)(a+2)\cdots(a+k-1).$$

These functions contain a surprising amount of arithmetic information. Most famously, Ramanujan's rapidly convergent hypergeometric formulas for $1/\pi$, such as

$${}_4F_3\left[\begin{matrix} \frac{7}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{6} & 1 & 1 \end{matrix} \; ; \; \frac{1}{4}\right] = \frac{4}{\pi},$$

are the theoretical basis for the Chudnovsky algorithm which has been utilized for all recent record-breaking computations of the digits of π. Hypergeometric functions also arise naturally in geometry as period integrals of certain Calabi–Yau manifolds, which are expected to be modular in the sense of the modularity theorem for elliptic curves. This yields connections between hypergeometric functions and modular forms.

Specifically, we often have a congruence between truncated hypergeometric functions and Fourier coefficients of modular forms. For a fixed prime p, let $F(\alpha,\beta;z)_{p-1}$ denote the truncation of the hypergeometric sum on the right-hand side of (2) at $p-1$. Under certain arithmetic conditions on α and β and for certain choices of $z=\lambda$, one often finds

$$F(\alpha,\beta;\lambda) \equiv a_p(f) \pmod{p},$$

where $a_p(f)$ is the p^{th} coefficient in the Fourier expansion of some modular form f. In certain circumstances, this congruence holds modulo higher powers of p. These *supercongruences* have immediate number theoretic utility for computing Fourier expansions of modular forms via the Weil bounds, and through connections to Calabi–Yau manifolds and Galois representation yield geometric and algebraic interpretations as well. In section §2, I expand upon and discuss my contributions to the theory of supercongruences.

I am intrigued by areas of number theory outside of modular forms as well. In §5, I discuss a project in which my collaborators and I are bridging two questions from arithmetic statistics and arithmetic geometry by investigating asymptotic bounds for the number of number fields, ordered by absolute discriminant, which are generated by algebraic points on a fixed plane curve.

2. Hypergeometric Supercongruences

2.1. **Background and motivation.** As discussed in §1, supercongruences refer to a stronger than expected relationship between values of truncated hypergeometric functions and some other arithmetic quantity, we will focus on the case where these quantities are Fourier coefficients of modular forms. Generally, such a supercongruence will have the form

(3)
$$p^m {}_nF_{n-1}\left[\begin{matrix} r_1 & r_2 & \cdots & r_n \\ & q_1 & \cdots & q_{n-1} \end{matrix} ;\ \lambda\right]_{p-1} := p^m \sum_{k=0}^{p-1} \frac{(r_1)_k \cdots (r_n)_k}{(q_1)_k \cdots (q_{n-1})_k k!} \equiv p^m \chi(p) a_p(f) \pmod{p^{1+s}},$$

where χ is some Dirichlet character. The p^m terms are necessary for clearing denominators in cases where the truncated hypergeometric series are not p-adically integral, and the integer s is referred to as the *depth* of the supercongruence. As a motivating example, Long, Tu, Yui, and Zudilin [27] obtain 14 depth 2 supercongruences with $m=0$, χ the trivial character, and for all primes $p>5$. These involve the fourteen hypergeometric functions

$${}_4F_3\left[\begin{matrix} r_1 & 1-r_1 & r_1 & 1-r_2 \\ & 1 & 1 & 1 \end{matrix} ;\ 1\right]_{p-1}$$

where $r_1, r_2 \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$ or $(r_1,r_2) \in \left\{(\frac{1}{5},\frac{2}{5}), (\frac{1}{8},\frac{3}{8}), (\frac{1}{10},\frac{3}{10}), (\frac{1}{12},\frac{5}{12})\right\}$ and certain modular forms of weight 4. These supercongruences were first observed by Rodriguez Villegas [32], and arise from families of rigid Calabi–Yau threefolds.

2.2. **Results.** In my dissertation, I extend the work of Long, Tu, Yui, and Zudilin to the following supercongruences, which were conjectured by Long:

Theorem 1 (Allen, [3])**.** *Assuming some expected conditions on the p-adic convergence of ratios of the truncations of the given hypergeometric series,*

$$p \cdot {}_4F_3 \begin{bmatrix} r_1 & 1-r_1 & r_2 & 1-r_2 \\ & 1 & q & 2-q \end{bmatrix} ; \; 1 \Big]_{p-1} \equiv \chi_{(r_1,r_2,q)}(p) a_p(f_{\{r_1,r_2,q\}}) \pmod{p^3}.$$

where $(r_1, r_2, q) \in \left\{(\frac{1}{2}, \frac{1}{2}, \frac{4}{3}), (\frac{1}{2}, \frac{1}{2}, \frac{5}{4}), (\frac{1}{2}, \frac{1}{2}, \frac{7}{6}), (\frac{1}{2}, \frac{1}{3}, \frac{5}{4}), (\frac{1}{2}, \frac{1}{3}, \frac{7}{6}), (\frac{1}{4}, \frac{3}{4}, \frac{7}{6})\right\}$*, each* $f_{r_1,r_2,q}$ *is an explicit weight 4 modular form and* $\chi_{(r_1,r_2,q)}$ *is either trivial or a quadratic character.*

These are the first supercongruences I know of to be worked on which involve parameters other than 1 appearing in the denominator, which necessitates a much more delicate p-adic analysis.

2.3. **Future and ongoing work.** The complete list of supercongruences which have been proved remains relatively small, but there are many more that have been observed numerically. Currently, I am working with Ling Long and Fang-Ting Tu to investigate supercongruences where the hypergeometric function is evaluated not at 1 but instead at CM (complex multiplication) points on modular curves. As an example, it appears that

$${}_2F_1 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{bmatrix} ; \; 2 \Big]_{p^s-1} \Big/ {}_2F_1 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{bmatrix} ; \; 2 \Big]_{p^{s-1}-1} \equiv -\left(\frac{2}{p}\right) \Gamma_p\left(\frac{1}{2}\right) \Gamma_p\left(\frac{1}{4}\right)^2 \mod p^{2s}.$$

We are developing a new approach for these types of supercongruences, where we consider the generating function of the truncated hypergeometric values

$$\sum_{n=0}^{\infty} F(\alpha, \beta; \lambda)_n x^n$$

and let $x = x(q)$ be a modular function. This transforms this generating function into a non-holomorphic modular form, where we allow for a single pole at the CM point λ. The desired supercongruence can then be cast in terms of Atkin–Swinnerton-Dyer congruences for the Fourier coefficients of the resulting form.

Additionally, Brian Grove, Long, Tu, and I have been investigating a number of hypergeometric evaluations and supercongruences for ${}_3F_2$ hypergeometric functions recently conjectured by Dawsey and McCarthy [12], which were initially found in connection to Paley graphs. These cases introduce a new obstacle as the parameters are not defined over $\mathbb{Q}$, meaning that $\{r_i\} \not\equiv \{-r_i\} \pmod{\mathbb{Z}}$. This condition restricts the primes for which the supercongruences can hold to a particular arithmetic progression, but we believe that the techniques of [27] and [3] can be adapted to this setting. Moreover, we are working to lift these supercongruences which will enable us to use the Jacquet-Langlands correspondence to give a geometric explanation of the supercongruences in Theorem 1.

3. Quantum modular forms and colored Jones polynomials

3.1. **Background and motivation.** One of the more recent entrants into the ever-growing list of types of modular forms are *quantum modular forms* [35], which are functions on rationals rather than on the upper-half plane. As $\mathbb{Q}$ is discrete, the modular transformation (1) and analyticity conditions must be modified. Specifically, we require

$$f\left(\frac{ax+b}{cx+d}\right) - (cx+d)^k f(\tau) = h(x)$$

to extend to a function on $\mathbb{R}$ with nice analytic properties. This definition is kept intentionally broad so as to encompass a large number of examples. If we instead use the disk model of hyperbolic space, such a function will be defined as a function on roots of unity via the map $\tau \to q = e^{2\pi\tau}$; this is the formulation we use below.

Quantum modular forms arise naturally as colored Jones polynomials—a key invariant arising in knot theory [22]. A result of Habiro [18] states that the N-colored Jones polynomial of a knot K always has a cyclotomic expansion

$$J_N(K;q) = \sum_{n=0}^{\infty} C_n(K;q)(q^{1+N};q)_n(q^{1-N};q)_n,$$

where $(a;q)_n$ denotes the q-Pochhammer symbol $(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$. Hikami and Lovejoy [19] show that for the family of torus knots $T_{(2,2t+1)}$ and any N^{th} root of unity ζ_N, $J_N(T_{(2,2t+1)};\zeta_N) = F_t(\zeta_N)$, where

$$F_t(q) := q^t \sum_{k_t \geq \cdots \geq k_1 \geq 0} (q;q)_{k_t} \prod_{i=1}^{t-1} q^{k_i(k_i+1)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_q$$

is a quantum modular form, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

is the q-binomial coefficient. The Quantum Modularity Conjecture of Zagier [16, 35] predicts this phenomenon occurs for all knots, a result which would imply the famous Volume Conjecture of Kashaev [23]. Additionally, Hikami and Lovejoy express the colored Jones polynomial of the mirror $J_N(T^*_{(2,2t+1)};q)$ in connection to a mixed mock modular form $U_t(x;q)$. The duality between the colored Jones polynomial of a knot and its mirror then leads to a duality between these different flavors of modularity.

3.2. **Ongoing and future work.** In an ongoing collaboration with Leah Sturman, we are investigating the modularity properties of the colored Jones polynomials of double twist knots. Let $K_{m,p}$ denote the double-twist knot obtained by linking ends of two twisted loops of $2m$ and $2p$ half-turns. Lovejoy and Osburn [26] find q-series $F_{m,p}(q)$ and $U_{m,p}(x;q)$ which correspond to the colored Jones polynomial of $K_{m,p}$ and its mirror. The modularity properties of these series remains unknown, although much work has been done recently by authors such as Borozonets [10] and Mortenson–Zwegers [31] to establish modularity of similar q-series. We aim to use a particular 2-parameter family of Bailey pairs associated to the knots $K_{m,p}$ to express the series $U_{m,p}(x;q)$ in terms of Hecke double sums and Appell–Lerch sums, which have been frequently used to establish modularity properties of hypergeometric q-series.

4. Holomorphic projection of sesquiharmonic forms

4.1. **Background and motivation.** In this section, we work with *sesquiharmonic forms*, which are modular forms where our analyticity and growth conditions are that f be real analytic and that the growth towards cusps is at most linear exponential. We also have the extra condition that f is annihilated by the differential operator $\xi_k \circ \Delta_k$, where ξ_k is the Brunier–Funke or shadow operator

$$\xi_k := iy^k \overline{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)},$$

and Δ_k is the weight k hyperbolic Laplacian

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \xi_{2-k} \circ \xi_k.$$

Every holomorphic cusp form is a sesquiharmonic form, and so we can associate a cusp form $\pi_k^{\mathrm{hol}}(f)$ to each sesquiharmonic form f by computing an orthogonal projection—with respect to the Petersson inner product—onto $S_k(\Gamma)$. This idea has been utilized by several authors in studying harmonic Maass forms [1, 6, 7, 8, 21, 29, 30].

4.2. **Results.**

Theorem 2 (A., Beckwith, Sharma, [4])**.** *Given a weight* $1/2$ *sesquiharmonic form* f *and a weight* $3/2$ *theta function* θ_χ *corresponding to an odd Dirichlet character* χ*, we compute the holomorphic projection of the weight* 2 *form* $f \cdot \theta_\chi$*.*

The exact formula is quite long, and so we omit it for brevity. As an application, we investigate the holomorphic projection of the product of a θ function with a particular sesquiharmonic form due to Duke–Imamoğlu–Tóth [13] and Ahlgren–Andersen–Samart [2], whose coefficients are given in terms of Hurwitz class numbers and a real quadratic generalization thereof which involves regulators and class numbers of real quadratic orders. In so doing, we find symmetry and structure in these class numbers utilizing the inherit structure on spaces of cusp forms.

4.3. **Future directions.** Most immediately, the Fourier expansion we currently have for our holomorphic projection involves a limit of an infinite sum which does not converge conditionally, but does appear to converge absolutely. If we could make this term explicit—which we believe could be done using multiple Dirichlet series and shifted convolution sums—then we would be able to deduce a lot more about the coefficients of a given sesquiharmonic form from the holomorphic projection. Next, the product $f(\tau)\theta_\chi(\tau)$ we utilize can also be thought of as the zeroth Rankin–Cohen bracket; it may be more fruitful to use higher Rankin–Cohen brackets instead. In particular, this would increase the weight of our sesquiharmonic form past $k = 2$, and would make the convergence of certain integrals appearing in Gross and Zagier's [17] formula for the coefficients in the holomorphic projection easier to determine. Finally, the theory of holomorphic projection could naturally be applied to other families of sesquiharmonic forms or extended more generally to polyharmonic Maass forms.

5. Number fields generated by plane curves

5.1. **Background and Motivation.** The following project is in collaboration with Renee Bell, Robert Lemke Oliver, Allechar Serrano López, and Tian An Wong, and originated at the first *Rethinking Number Theory* workshop in October 2020.

Let K be a number field and $\mathscr{C}$ a smooth plane curve of genus $g \geq 2$. A theorem of Faltings [14, 15] states that the set of K-rational points of $\mathscr{C}$ is always finite. However, the number of fields generated by $\mathscr{C}$ over K need not be finite. That is, there may be infinitely many field extensions $K(P)$ over K that contain a point P on our curve $\mathscr{C}$ which is contained in no intermediate extension of K. In their program on Diophantine stability [28], Mazur and Rubin suggest considering the family of all fields generated by a given curve as a tool to study the curve $\mathscr{C}$ itself.

Motivated by this idea, multiple authors have considered the set of fields generated by a fixed curve through the lens of arithmetic statistics. Formally, we let $\mathscr{C}(\bar{\mathbb{Q}})$ denote the set of all algebraic points on the curve $\mathscr{C}$, and we define

$$\mathcal{F}_K(\mathscr{C}) := \left\{K(P) : P \in \mathscr{C}(\bar{\mathbb{Q}})\right\}.$$

We then define the quantity

$$N_n^{\mathscr{C}}(X) := \#\left\{L/K \mid L \in \mathcal{F}_K(\mathscr{C}) \ : \ L = K(P), [L : K] = n, |\mathrm{Disc} L/K| \leq X\right\}.$$

Asymptotic lower bounds for these functions have been found for elliptic curves by Lemke Oliver and Thorne [25], for hyperelliptic curves by Keyes [24], and for superelliptic curves by Beneish and Keyes [9]. The goal of this project is to find similar asymptotic lower bounds that hold for all smooth plane curves $\mathscr{C}$.

5.2. **Results.** With notation as above, we show the following:

Theorem 3 (A., Bell, Lemke Oliver, Serrano López, Wong, [5])**.** *Let $\mathscr{C} : f(x, y) = 0$ be a nonsingular, absolutely irreducible plane curve over $\mathbb{Q}$. Let $m := \gcd\{\deg_x f, \deg_y f\}$. There exists $n_0 \in \mathbb{Z}$ such that if $n \geq n_0$ is a multiple of m, then there is a constant $c_n > 0$ depending only on n and $\mathscr{C}$ satisfying $c_n = \frac{1}{m^2} + o_{n\to\infty}(1)$ such that*

$$N_n^{\mathscr{C}}(X) \gg_{\mathscr{C},n} X^{c_n}$$

as $X \to \infty$, where c_n is computable and approaches $\frac{1}{\min\left\{\deg_x(f), \deg_y(f)\right\}^2}$ as $n \to \infty$.

Our approach is to intersect our curve $\mathscr{C}$ with a parameterized curve $g(t) = \langle x(t), y(t)\rangle$ where $x(t)$ and $y(t)$ are rational functions with integer coefficients. These intersect exactly when $f(x(t), y(t)) = 0$, which occurs exactly when $t = \alpha$ is an algebraic number, namely the root of the numerator of the rational function $f(x(t), y(t))$. Assuming that this numerator is irreducible, this gives us a number field $\mathbb{Q}(\alpha)$ of degree n which contains a point $(x(\alpha), y(\alpha))$ lying on our curve. As we run through all such parametric curves $\langle x(t), y(t)\rangle$, this process generates infinitely many algebraic points on $\mathscr{C}$ and hence infinitely many number fields generated by $\mathscr{C}$. Of course, multiple parametric curves can intersect $\mathscr{C}$ at a given point, and so much of the work in determining our asymptotic lower bound comes from determining redundancy in the number fields we count in this way.

5.3. **Future directions.** We are currently working to extend this result to count only number fields whose Galois closure has Galois group S_n. Intuitively, we should have a nearly identical asymptotic lower bound, as we would expect any random irreducible polynomial we write down to have Galois group S_n. However, proving this in the full generality of our setting is quite difficult. Additionally, we are currently working to apply this result to a result on number fields where the Jacobian $J(\mathscr{C})$ grows in rank, analogous to what Lemke Oliver and Thorne show for elliptic curves [25].

We are only scraping the surface of questions that could be asked about fields generated by curves, and so there are many directions I hope to continue this collaboration in, such as considering Galois groups which are not S_n, searching for asymptotic upper bounds, or considering more general geometric objects than only algebraic plane curves, such as algebraic hypersurfaces.

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